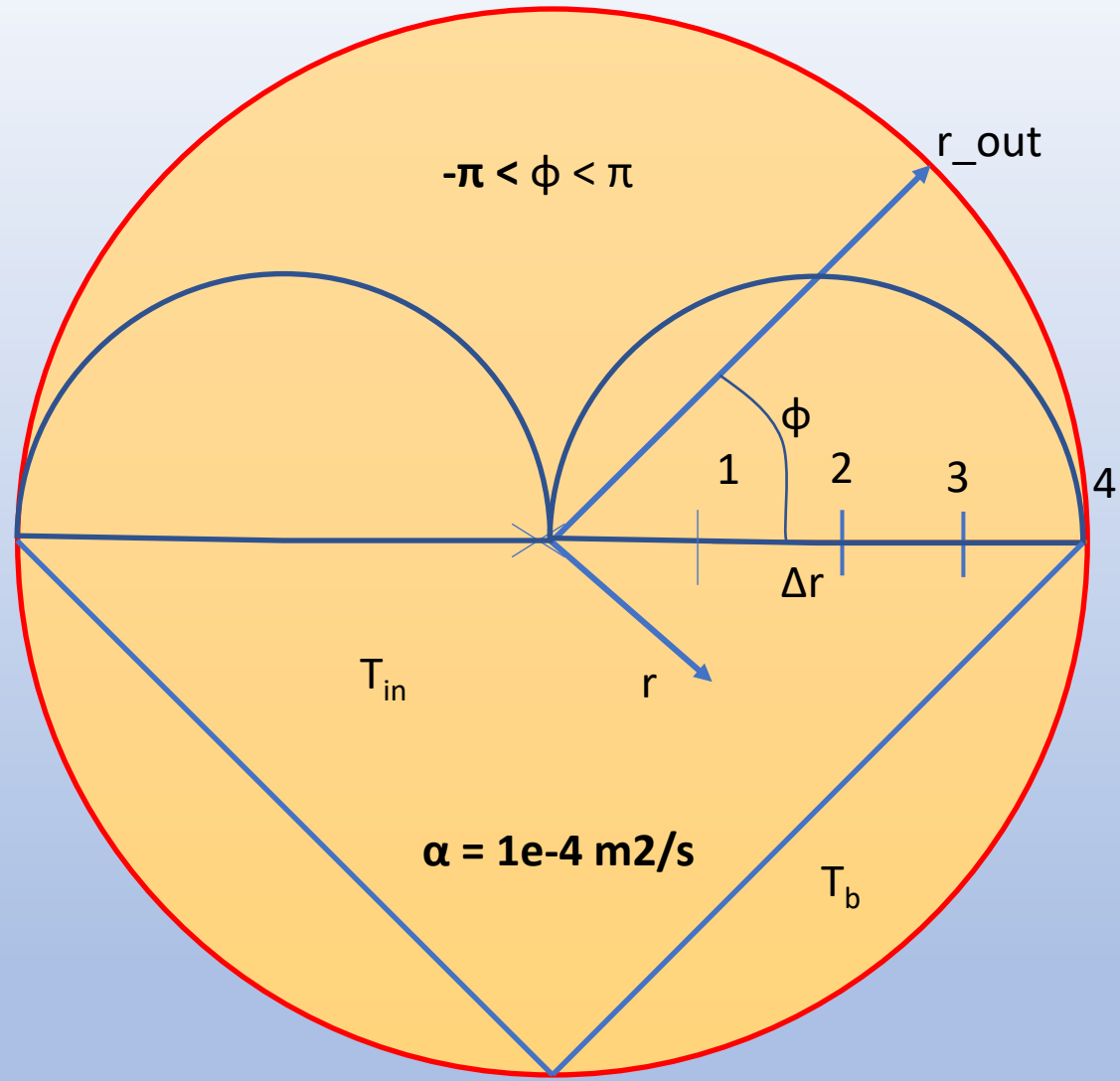


Solve 2D Transient Heat Conduction Problem  
in Cylindrical Coordinates Using FTCS  
Finite Difference Method

## Objectives

- Present a simple 2D heat conduction problem in cylindrical coordinates
- Temperature variation is along the radial direction and along the phase angle ( $\phi$ )
- Solve the problem using FTCS FDM
- Vary grid spacings and obtain solutions using FDM and present the results graphically



2D DOMAIN

# General Heat Conduction Equation in 3D Cylindrical Coordinates

- $$\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r}\right) \left(\frac{\partial T}{\partial r}\right) + \left(\frac{1}{r^2}\right) \left(\frac{\partial^2 T}{\partial \Phi^2}\right) + \frac{\partial^2 T}{\partial z^2} + \frac{g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \dots\dots\dots(1) ;$$

- $T = T(r, \phi, z, t)$ ;  $r$  – radial dimension,  $\phi$  – phase angle,  $z$  – vertical dimension,  $t$  – time,  $s$
- $\alpha$  – Thermal diffusivity,  $\frac{m^2}{s}$
- $\alpha = k / (\rho * c)$
- $k$  – thermal conductivity of the material ,  $W/(m K)$
- $\rho$  - density of the material,  $kg/m^3$
- $c$  – specific heat capacity of the material,  $J/(kg K)$
- $g$  – volumetric rate of internal heat generation,  $W/(m^3)$
- We assume that  $k$  is uniform in the domain (homogeneous and isotropic)

For 2D transient heat conduction along radial direction and along the phase angle, with no heat generation, equation (1) reduces to a simpler form

- We assume temperature does not vary significantly in z direction when compared with radial direction and phase angle e.g., top and bottom surfaces are insulated.
- $g = 0$  (no heat generation);
- $\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r}\right) \left(\frac{\partial T}{\partial r}\right) + \left(\frac{1}{r^2}\right) \left(\frac{\partial^2 T}{\partial \Phi^2}\right) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$  .....(2);  $T = T(r, \phi, t)$ ;
- IC:  $T(r, \phi, t=0) = T_{in} = 1000 \text{ deg.C}$ ; (at the interior nodes)
- BCs:  $r \geq r_{limit}, T(r, \phi, t) = T_b = 0 \text{ deg.C}$ ;  $t \geq 0$ ;
- To obtain  $T(r, \phi, t)$  we need to solve the above PDE.
- We will utilize Finite Difference Method to solve the above PDE.
- To do so, we need to replace the partial derivatives with finite difference approximations
- We will replace both the space ( $r, \phi$ ) derivatives with centered difference approximations and the time ( $t$ ) derivative with first order forward difference approximation.

- Let  $i$  represent the node location along the radial direction and  $j$  represent the node location along the phase angle. Then,

$$\bullet \left( \frac{T_{i-1,j}^n - 2 * T_{i,j}^n + T_{i+1,j}^n}{\Delta r^2} \right) + \left( \frac{1}{r_i} \right) \left( \frac{T_{i+1,j}^n - T_{i-1,j}^n}{2 * \Delta r} \right) + \left( \frac{1}{r_i^2} \right) \left( \frac{T_{i,j-1}^n - 2 * T_{i,j}^n + T_{i,j+1}^n}{\Delta \phi^2} \right)$$

$$= \frac{1}{\alpha} \left( \frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} \right)$$

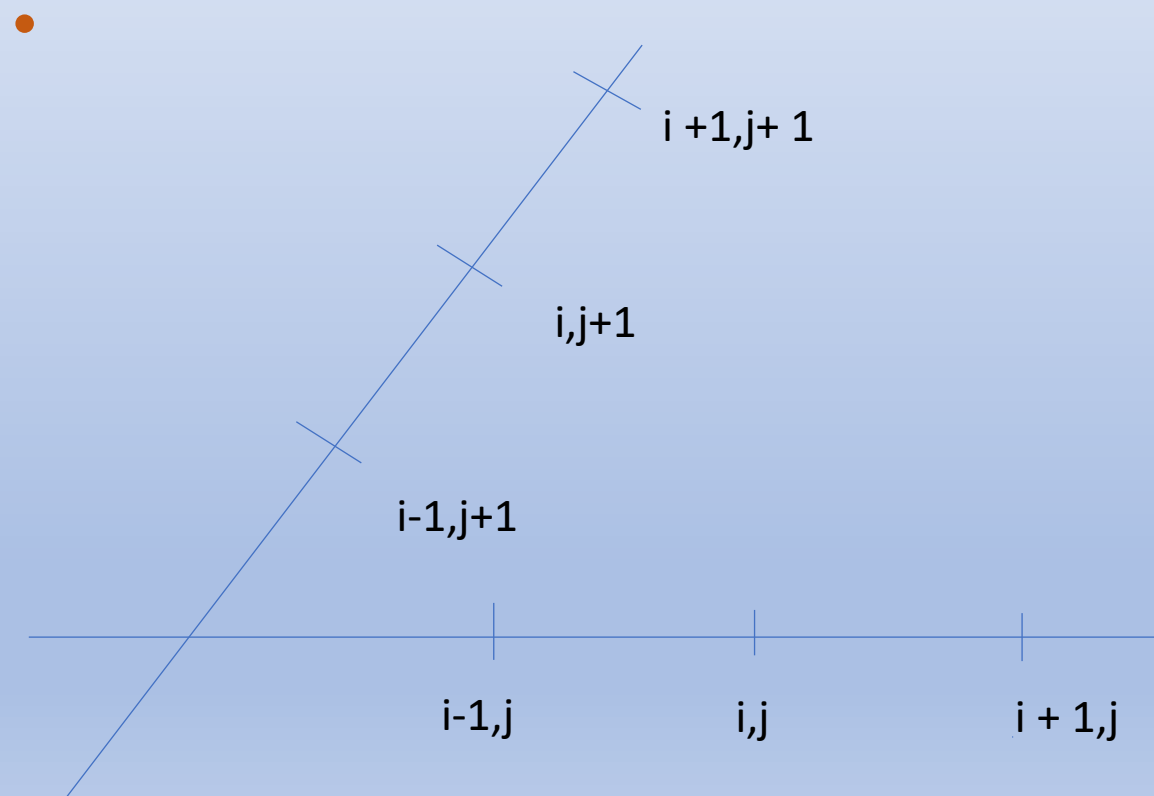
$$\bullet T_{i,j}^{n+1} = T_{i,j}^n + (\alpha * \Delta t) * \left( \left( \frac{T_{i-1,j}^n - 2 * T_{i,j}^n + T_{i+1,j}^n}{\Delta r^2} \right) + \left( \frac{1}{r_i} \right) \left( \frac{T_{i+1,j}^n - T_{i-1,j}^n}{2 * \Delta r} \right) + \left( \frac{1}{r_i^2} \right) \left( \frac{T_{i,j-1}^n - 2 * T_{i,j}^n + T_{i,j+1}^n}{\Delta \phi^2} \right) \right)$$

- $T_{i,j}^{n+1} = T_{i,j}^n + \left(\frac{\alpha * \Delta t}{\Delta r^2}\right) * \left( (T_{i-1,j}^n - 2 * T_{i,j}^n + T_{i+1,j}^n) + \left(\frac{\Delta r}{2 * r_i}\right) (T_{i+1,j}^n - T_{i-1,j}^n) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (T_{i,j-1}^n - 2 * T_{i,j}^n + T_{i,j+1}^n) \right)$

- Let  $d = \left(\frac{\alpha * \Delta t}{\Delta r^2}\right)$ , we then get

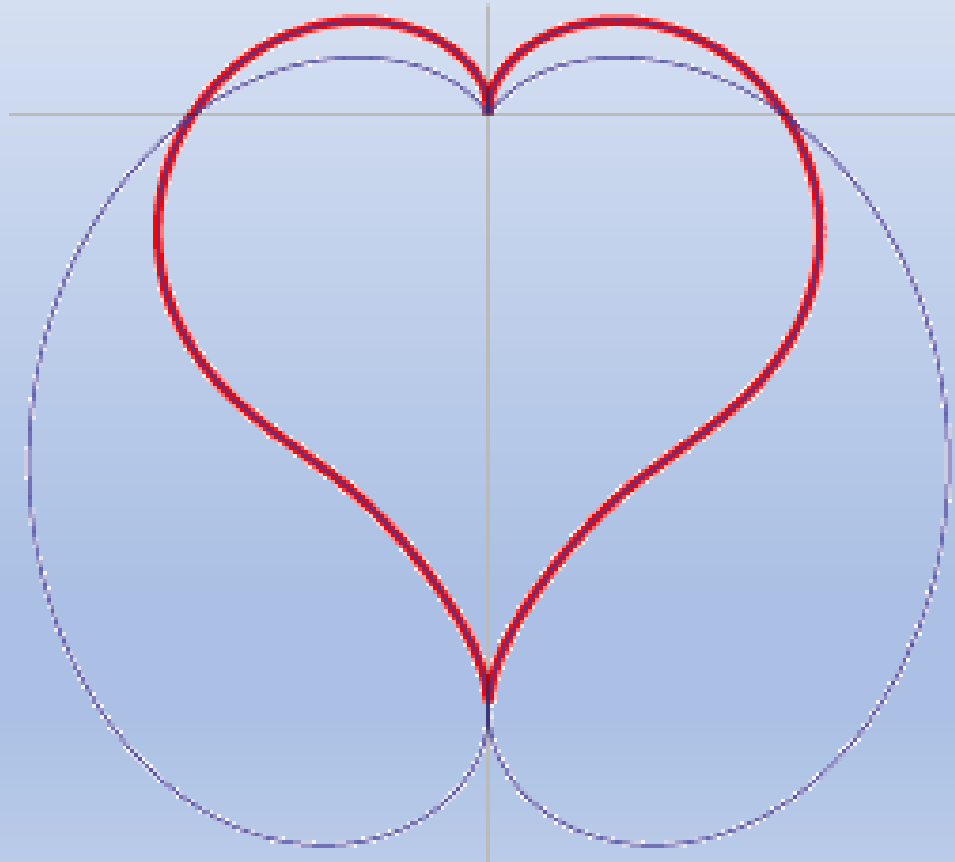
- $T_{i,j}^{n+1} = T_{i,j}^n + d * \left( (T_{i-1,j}^n - 2 * T_{i,j}^n + T_{i+1,j}^n) + \left(\frac{\Delta r}{2 * r_i}\right) (T_{i+1,j}^n - T_{i-1,j}^n) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (T_{i,j-1}^n - 2 * T_{i,j}^n + T_{i,j+1}^n) \right) \dots\dots\dots(3)$

- Equation (3) is the finite difference approximation of the original PDE we were trying to solve.
- Here  $i, j$  represent the node location on the discretized domain.
- The finite difference stencil is given below.



## Boundary Conditions:

- One particular heart equation in polar coordinates is given as:
- $r_{\text{limit}}(\phi) = 2 - 2 \cdot \sin(\phi) + (\sin(\phi) \cdot \sqrt{\text{abs}(\cos(\phi))}) / (\sin(\phi) + 1.4)$



Boundary Conditions (contd):

- If  $r \geq r\_limit$ , for any  $\phi$ ,  $T(r,\phi) = T_b$  .....(4)

- At  $r = 0$ , for all  $\phi$ ,  $\left(\frac{\partial T}{\partial r}\right) = 0$ .

- Applying forward difference approximation,

- $\left(\frac{T_{i+1,j}^n - T_{i,j}^n}{\Delta r}\right) = 0$ , for  $i = 0$  for all  $j$ ;

- $T_{0,j} = T_{1,j}$  .....(5)

- Inside the heart geometry, eq (3) is applicable, that is,

- If  $r < r\_limit$ , for all  $\phi$ ,  $T(r,\phi) = T_{i,j}^{n+1}$  (see below)

- $T_{i,j}^{n+1} = T_{i,j}^n + d * \left( (T_{i-1,j}^n - 2 * T_{i,j}^n + T_{i+1,j}^n) + \left(\frac{\Delta r}{2 * r_i}\right) (T_{i+1,j}^n - T_{i-1,j}^n) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (T_{i,j-1}^n - 2 * T_{i,j}^n + T_{i,j+1}^n) \right)$  .....(3)

- Stability Criteria (Based on von Neumann Method)

- $f_{i+/-1}^n = f_i^n * e^{+/-I * (k_m * \Delta x)}$  ;  $k_m$  - wave number =  $\left(\frac{2 * \pi * m}{2 * L}\right)$  ;  $m = 1... \infty$ ;  $I = \sqrt{-1}$

- $f_{i+/-1}^n = f_i^n * e^{+/-I * (\theta)}$  ;  $\theta = (k_m * \Delta x)$

- $f_i^{n+1} = G * f_i^n$

- Amplification factor  $|G| \leq 1$

- $\cos \theta = \left(\frac{e^{I\theta} + e^{-I\theta}}{2}\right)$  ;  $\sin \theta = \left(\frac{e^{I\theta} - e^{-I\theta}}{2I}\right)$

- Stability Criteria (Based on von Neumann Method)

- $$T_{i,j}^{n+1} = T_{i,j}^n + d * \left( (T_{i-1,j}^n - 2 * T_{i,j}^n + T_{i+1,j}^n) + \left(\frac{\Delta r}{2 * r_i}\right) (T_{i+1,j}^n - T_{i-1,j}^n) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (T_{i,j-1}^n - 2 * T_{i,j}^n + T_{i,j+1}^n) \right) \dots \dots \dots (3)$$

- $$T_{i,j}^{n+1} = T_{i,j}^n + d * \left( (T_{i,j}^n * e^{-I * (k_m * \Delta r)} - 2 * T_{i,j}^n + T_{i,j}^n * e^{I * (k_m * \Delta r)}) + \left(\frac{\Delta r}{2 * r_i}\right) (T_{i,j}^n * e^{I * (k_m * \Delta r)} - T_{i,j}^n * e^{-I * (k_m * \Delta r)}) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (T_{i,j}^n * e^{-I * (k_m * \Delta \phi)} - 2 * T_{i,j}^n + T_{i,j}^n * e^{I * (k_m * \Delta \phi)}) \right)$$

- Stability Criteria (Based on von Neumann Method)

- Let  $\theta_1 = (k_m * \Delta r)$  ;  $\theta_2 = (k_m * \Delta \phi)$  ;

- $$T_{i,j}^{n+1} = T_{i,j}^n + d * \left( (T_{i,j}^n * e^{-I * (\theta_1)} - 2 * T_{i,j}^n + T_{i,j}^n * e^{I * (\theta_1)}) + \left( \frac{\Delta r}{2 * r_i} \right) (T_{i,j}^n * e^{I * (\theta_1)} - T_{i,j}^n * e^{-I * (\theta_1)}) + \left( \frac{1}{r_i^2} \right) * \left( \frac{\Delta r^2}{\Delta \phi^2} \right) * (T_{i,j}^n * e^{-I * (\theta_2)} - 2 * T_{i,j}^n + T_{i,j}^n * e^{I * (\theta_2)}) \right)$$

- Stability Criteria (Based on von Neumann Method)

- $$T_{i,j}^{n+1} = T_{i,j}^n \left( 1 + d * \left( \left( e^{-I * (\theta 1)} - 2 + e^{I * (\theta 1)} \right) + \left( \frac{\Delta r}{2 * r_i} \right) \left( e^{I * (\theta 1)} - e^{-I * (\theta 1)} \right) + \left( \frac{1}{r_i^2} \right) * \left( \frac{\Delta r^2}{\Delta \phi^2} \right) * \left( e^{-I * (\theta 2)} - 2 + e^{I * (\theta 2)} \right) \right) \right)$$

- $$T_{i,j}^{n+1} = T_{i,j}^n \left( 1 + d * \left( \left( 2 * \cos(\theta 1) - 2 \right) + \left( \frac{\Delta r}{2 * r_i} \right) \left( 2 * I * \sin(\theta 1) \right) + \left( \frac{1}{r_i^2} \right) * \left( \frac{\Delta r^2}{\Delta \phi^2} \right) * \left( 2 * \cos(\theta 2) - 2 \right) \right) \right)$$

- Stability Criteria (Based on von Neumann Method)

- $$G = 1 + d * \left( 2 * (\cos(\theta_1) - 1) + \left(\frac{\Delta r}{r_i}\right) * (\sin(\theta_1)) * I + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * 2 * (\cos(\theta_2) - 1) \right)$$

- $$G = 1 + d * \left( 2 * (\cos(\theta_1) - 1) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * 2 * (\cos(\theta_2) - 1) \right) + \left( d * \left(\frac{\Delta r}{r_i}\right) * (\sin(\theta_1)) * I \right)$$

- Stability Criteria  $|G| \leq 1$

- Stability Criteria (Based on von Neumann Method)

- $$\sqrt{\left(1 + 2 * d * \left( (\cos(\theta_1) - 1) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (\cos(\theta_2) - 1) \right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (\sin(\theta_1))\right)^2} \leq 1$$

- $-1 \leq \cos(\theta_1) \leq 1; -1 \leq \cos(\theta_2) \leq 1; -1 \leq \sin(\theta_1) \leq 1;$

- Case 1 :  $\cos(\theta_1) = 1; \cos(\theta_2) = 1; \sin(\theta_1) = 0;$

- $$G1 = \sqrt{\left(1 + 2 * d * \left( (1 - 1) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (1 - 1) \right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (0)\right)^2}$$

- $G1 = 1$

- Stability Criteria (Based on von Neumann Method)

- $$\sqrt{\left(1 + 2 * d * \left(\cos(\theta_1) - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(\cos(\theta_2) - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * \sin(\theta_1)\right)^2} \leq 1$$

- Case 2 :  $\cos(\theta_1) = -1$ ;  $\cos(\theta_2) = 1$ ;  $\sin(\theta_1) = 0$ ;

- $$G2 = \sqrt{\left(1 + 2 * d * \left(-1 - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(1 - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (0)\right)^2}$$

- $$G2 = \sqrt{(1 - 4 * d)^2}$$

- Stability Criteria (Based on von Neumann Method)

- $$\sqrt{\left(1 + 2 * d * \left(\cos(\theta_1) - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(\cos(\theta_2) - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * \sin(\theta_1)\right)^2} \leq 1$$

- Case 3 :  $\cos(\theta_1) = 0$ ;  $\cos(\theta_2) = 1$ ;  $\sin(\theta_1) = 1$ ;

- $$G3 = \sqrt{\left(1 + 2 * d * \left(0 - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (1 - 1)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (1)\right)^2}$$

- $$G3 = \sqrt{(1 - 2 * d)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right)\right)^2}$$

- Stability Criteria (Based on von Neumann Method)

- $$\sqrt{\left(1 + 2 * d * \left(\cos(\theta_1) - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(\cos(\theta_2) - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * \sin(\theta_1)\right)^2} \leq 1$$

- Case 4 :  $\cos(\theta_1) = 0$ ;  $\cos(\theta_2) = 1$ ;  $\sin(\theta_1) = -1$ ;

- $$G4 = \sqrt{\left(1 + 2 * d * \left(0 - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(1 - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (-1)\right)^2}$$

- $$G4 = \sqrt{(1 - 2 * d)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right)\right)^2}$$
 ; Note  $G4 = G3$

- Stability Criteria (Based on von Neumann Method)

- $$\sqrt{\left(1 + 2 * d * \left( (\cos(\theta_1) - 1) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (\cos(\theta_2) - 1) \right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (\sin(\theta_1))\right)^2} \leq 1$$

- Case 5 :  $\cos(\theta_1) = 1$ ;  $\cos(\theta_2) = -1$ ;  $\sin(\theta_1) = 0$ ;

- $$G5 = \sqrt{\left(1 + 2 * d * \left( (1 - 1) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (-1 - 1) \right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (0)\right)^2}$$

- $$G5 = \sqrt{\left(1 - 4 * d * \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right)\right)^2}$$

- Stability Criteria (Based on von Neumann Method)

- $$\sqrt{\left(1 + 2 * d * \left(\cos(\theta_1) - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(\cos(\theta_2) - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * \sin(\theta_1)\right)^2} \leq 1$$

- Case 6 :  $\cos(\theta_1) = -1$ ;  $\cos(\theta_2) = -1$ ;  $\sin(\theta_1) = 0$ ;

- $$G6 = \sqrt{\left(1 + 2 * d * \left(-1 - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(-1 - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (0)\right)^2}$$

- $$G6 = \sqrt{\left(1 - 4 * d - 4 * d * \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right)\right)^2}$$

- Stability Criteria (Based on von Neumann Method)

- $$\sqrt{\left(1 + 2 * d * \left(\cos(\theta_1) - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(\cos(\theta_2) - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * \sin(\theta_1)\right)^2} \leq 1$$

- Case 7 :  $\cos(\theta_1) = 0$ ;  $\cos(\theta_2) = -1$ ;  $\sin(\theta_1) = 1$ ;

- $$G7 = \sqrt{\left(1 + 2 * d * \left(0 - 1\right) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * \left(-1 - 1\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (1)\right)^2}$$

- $$G7 = \sqrt{\left(1 - 2 * d - 4 * d * \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right)\right)^2}$$

- Stability Criteria (Based on von Neumann Method)

- $$\sqrt{\left(1 + 2 * d * \left(\cos(\theta_1) - 1 + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (\cos(\theta_2) - 1)\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (\sin(\theta_1))\right)^2} \leq 1$$

- Case 8 :  $\cos(\theta_1) = 0$ ;  $\cos(\theta_2) = -1$ ;  $\sin(\theta_1) = -1$ ;

- $$G8 = \sqrt{\left(1 + 2 * d * \left(0 - 1 + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right) * (-1 - 1)\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right) * (-1)\right)^2}$$

- $$G8 = \sqrt{\left(1 - 2 * d - 4 * d * \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta \phi^2}\right)\right)^2 + \left(d * \left(\frac{\Delta r}{r_i}\right)\right)^2} ; \text{ Note } G8 = G7$$

- Let  $\alpha = 1e-4 \text{ m}^2/\text{s}$ ,  $\Delta t = \left(\frac{1}{6}\right) \text{ s}$ ,  $\Delta r = 0.39 \text{ m}$ ,  $\Delta\phi = \left(\frac{\pi}{5}\right)$ , then
- $d = \left(\frac{\alpha * \Delta t}{\Delta r^2}\right) = \left(\frac{1e-4 * \left(\frac{1}{6}\right)}{0.39^2}\right) = 0.000109577$
- Verify Stability Condition using Von Neumann Stability Condition
- As an example, at time  $t=0$ ,
- Boundary temperature  $T_b = 0 \text{ }^\circ\text{C}$  (for all  $t \geq 0$ , i.e.,  $n \geq 0$ ),
- Initial temperature,  $T_{in} = T_{i,j}^0 = 1000 \text{ }^\circ\text{C}$
- Let  $n = 0$ ,  $r(10) = 2 \text{ m}$ , then
- $$T_{i,j}^{n+1} = T_{i,j}^n + d * \left( (T_{i-1,j}^n - 2 * T_{i,j}^n + T_{i+1,j}^n) + \left(\frac{\Delta r}{2 * r_i}\right) (T_{i+1,j}^n - T_{i-1,j}^n) + \left(\frac{1}{r_i^2}\right) * \left(\frac{\Delta r^2}{\Delta\phi^2}\right) * (T_{i,j-1}^n - 2 * T_{i,j}^n + T_{i,j+1}^n) \right)$$

- At the node adjacent to the boundary nodes, for example,

$$\bullet T_{10,1}^1 = T_{10,1}^0 + d * \left( (T_{9,1}^0 - 2 * T_{10,1}^0 + T_{11,1}^0) + \left( \frac{\Delta r}{2 * r_{10}} \right) (T_{11,1}^0 - T_{9,1}^0) + \left( \frac{1}{r_{10}^2} \right) * \left( \frac{\Delta r^2}{\Delta \phi^2} \right) * (T_{10,-1}^0 - 2 * T_{10,1}^0 + T_{10,2}^0) \right)$$

$$\bullet T_{10,1}^1 = 1000 + 0.000109577 * \left( (1000 - 2 * 1000 + 0) + \left( \frac{0.39}{2 * 2} \right) (0 - 1000) + \left( \frac{1}{2^2} \right) * \left( \frac{0.39^2}{\left( \frac{\pi}{5} \right)^2} \right) * (0 - 2 * 1000 + 1000) \right) = 1000 - 0.14136937 = 999.869 \text{ } ^\circ\text{C}$$

- Using the same procedure, we can find the temperatures at all the interior nodes at  $n = 1$
- Likewise, we can find the temperatures at all the interior nodes at the next time step by choosing  $n = 2$  and so on
- Graphical results are presented using MATLAB for this case.
- Using MATLAB or other software, we can develop codes for a general case where the number of grid spacings and time steps can be varied as desired.
- And solutions obtained accordingly.

# Summary

In this video,

- We presented a 2D transient heat conduction problem in Cylindrical Coordinates
- Temperature variation is along the radial direction and along the phase angle
- We solved the problem using FTCS Finite difference method and obtained the temperature profile
- We re-solved the problem using smaller grid spacings and presented the results
- In future videos, we can explore more challenging problems.